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On infinite divisibility of positive definite functions arising from operator means

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Abstract

For a (point-wisely non-negative) positive definite function a certain criterion for its infinite divisibility (i.e., all its fractional powers are also positive definite) is obtained. This criterion enables us to show infinite divisibility for many positive definite functions appearing naturally in study of operator means. In particular, we determine when the function

$$\frac{\cosh(\nu x) + s'}{\cosh x + s} \quad \big(\nu \in [0,1];\ s, s' \in (-1,1]\big)$$

is infinitely divisible.

Keywords: Hadamard product; Infinitely divisible function; Infinitely divisible matrix; Operator mean; Positive definite function

1. Introduction

Operator means and comparison of their (unitarily invariant) norms are under active investigation (see [5,12–14,17,22] for instance), where many positive matrices with non-negative entries

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naturally appear. Typical examples are $A = [a_{ij}]_{i,j=1,2,\ldots,n}$ and $B = [b_{ij}]_{i,j=1,2,\ldots,n}$ (or their suitable variants) with entries

$$a_{ij} = \frac{\lambda_i^\theta - \lambda_j^\theta}{\lambda_i - \lambda_j}, \qquad b_{ij} = \frac{\lambda_i^\theta + \lambda_j^\theta}{\lambda_i + \lambda_j} \quad (0 < \theta < 1)$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$, and the positivity of A, B corresponds to the positive definiteness of the functions $\sinh(\theta x)/\sinh x$ and $\cosh(\theta x)/\cosh x$, respectively (see Section 2 for details). Thanks to Horn's theorem [15] the positive matrix A is actually infinitely divisible in the sense that

$$\left[a_{ij}^r\right]_{i,j=1,2,\ldots,n} \geqslant 0$$

for each $r \in (0,1)$, showing that the positive definite function $f(x) = \sinh(\theta x)/\sinh x$ is indeed infinitely divisible (see (2) in Section 2), i.e., all the fractional powers $f(x)^r$ $(0 < r < 1)$ are positive definite. The reader is advised to see the recent (partly survey) article [1], where the importance of this concept is explained together with an abundance of old and new examples.

In our previous work [4] among other things the infinite divisibility of the positive definite function $\cosh(\theta x)/\cosh x$ $(0 < \theta < 1)$ was established based on a certain power series trick (see the power series (8) used in Section 4), which shows the infinite divisibility of the matrix

$$\left[\frac{\lambda_i^\theta + \lambda_j^\theta}{\lambda_i + \lambda_j}\right]_{i,j=1,2,\ldots,n}.$$

From this the infinite divisibility of many "mean matrices" can be derived (as was demonstrated in [4]), and the present work can be considered as a natural continuation to this study. Let us recall that for an operator monotone function $g(x) : [0,\infty) \to [0,\infty)$ the positivity

$$\left[\frac{g(\lambda_i) + g(\lambda_j)}{\lambda_i + \lambda_j}\right]_{i,j=1,2,\ldots,n} \geqslant 0$$

is known ([19], see also [5, Remark 5.2]). However, a general result on the infinite divisibility for matrices of this type (except for the special case $g(x) = x^\theta$) is unknown and seems to deserve investigation.

The classical Bochner theorem asserts that a function is positive definite if and only if its Fourier transform is non-negative. Difficulty for study on infinite divisibility lies in the fact that explicit computation for Fourier transforms of fractional powers of functions in question is almost hopeless. A notable exception is

$$\int_{-\infty}^{\infty} \frac{e^{ixy}\,dx}{\cosh^r x} = \frac{2^{r-1}|\Gamma((r+iy)/2)|^2}{\Gamma(r)}, \qquad r \in (0,1)$$

(see [10, 3.985 on p. 507] or [14, Appendix A.6]). This formula yields the positive definiteness of $1/\cosh^r x$, i.e., the infinite divisibility of $1/\cosh x$, which corresponds to the well-known fact that the Cauchy matrix

$$\left[\frac{1}{\lambda_i+\lambda_j}\right]_{i,j=1,2,\dots,n}$$

is infinitely divisible.

Based on some complex analysis technique we will obtain a certain criterion for infinite divisibility (see Theorem 2 and Corollary 3) in Section 3, and the Hadamard factorization theorem indeed plays a crucial role in our proof. Our criterion enables us to show the infinite divisibility for the above-mentioned functions (and many others) in a unified way. In Section 4, by combining this criterion with explicit computations of relevant Fourier transforms, we will determine when the function

$$\frac{\cosh(\nu x)+s'}{\cosh x+s} \quad \bigl(\text{with } \nu\in[0,1] \text{ and } s,s'\in(-1,1]\bigr)$$

is infinitely divisible. It is shown to be so exactly when the function is positive definite (see Theorem 11). Here is one of the most basic norm inequalities on operator means: for Hilbert space operators H, K, X with $H,K\geqslant 0$ and a unitarily invariant norm $|||\cdot|||$ we have

$$\bigl|\bigl|\bigl|H^{\theta}XK^{1-\theta}+H^{1-\theta}XK^{\theta}\bigr|\bigr|\bigr| \leqslant |||HX+XK||| \quad (0\leqslant\theta\leqslant 1).$$

It is known as the Heinz inequality [11], and can be derived from the positivity of the matrix B or equivalently from the positive definiteness of $\cosh(\theta x)/\cosh x$ (see [14] for instance). In recent years various estimates on norms of more general operator means such as

$$\bigl|\bigl|\bigl|H^{\theta}XK^{1-\theta}+H^{1-\theta}XK^{\theta}+xH^{1/2}XK^{1/2}\bigr|\bigr|\bigr|$$

(containing a parameter x) have been studied by several workers (see [5,19,22] for instance). Our analysis here gives rise to very precise information on such "generalized" Heinz-type norm inequalities. This subject (together with related topics) will be covered in our forthcoming article [18]. In the final Section 5 we will discuss infinite divisibility for miscellaneous functions such as

$$\frac{1}{\cosh z+s\cosh(\alpha z)} \quad \bigl(\text{with } \alpha\in[0,1] \text{ and } s\in(-1,1]\bigr)$$

(see Theorem 14).

2. Preliminaries

2.1. Infinitely divisible matrices

The classical Schur theorem states that the Hadamard product (or Schur product) $A\circ B$ of positive matrices A, B is positive: For $A=[a_{ij}]$, $B=[b_{ij}]\geqslant 0$ we have

$$A\circ B=[a_{ij}b_{ij}]\geqslant 0.$$

Here and throughout the positivity $A=[a_{ij}]_{i,j=1,2,\dots,n}\geqslant 0$ means

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\overline{\xi_j}\geqslant 0$$

for each $\xi_i\in\mathbf{C}$. In particular, if $A=[a_{ij}]$ is positive, then so are the Hadamard powers $A^{\circ m}=[a_{ij}^m]$, $m\in\mathbf{N}$. When each entry is non-negative in addition, fractional Hadamard powers $A^{\circ r}=[a_{ij}^r]$ $(r>0)$ also make sense. It is known (see [9, Theorem 2.2]) that (i) for such an $n\times n$ matrix we have $A^{\circ r}\geqslant 0$ as long as $r\geqslant n-2$ and (ii) this lower bound $n-2$ is optimal. Note that for a positive matrix $A=[a_{ij}]$ with $a_{ij}\geqslant 0$, the following three conditions are equivalent (thanks to the Schur theorem and the obvious continuity argument):

(i) $A^{\circ\frac{1}{m}}\geqslant 0$ for each $m\in\mathbf{N}$;
(ii) $A^{\circ r}\geqslant 0$ for each $r\in(0,1)$;
(iii) $A^{\circ r}\geqslant 0$ for each $r\in(0,\infty)$.

A matrix satisfying these conditions is called an *infinitely divisible* matrix. A very readable account on such matrices can be found in [16] and many examples are worked out in [1,4].

2.2. *Infinitely divisible functions*

A study on positive matrices is closely related to that of positive definite functions. Let us take $f(x)=\sinh(\theta x)/\sinh x$ with $\theta\in(0,1)$ for instance and observe

$$\begin{aligned}\sum_{i,j=1}^{n} f\left(\frac{1}{2}\log\lambda_i-\frac{1}{2}\log\lambda_j\right)\xi_i\overline{\xi_j}&=\sum_{i,j=1}^{n}\frac{((\lambda_i/\lambda_j)^{\theta/2}-(\lambda_j/\lambda_i)^{\theta/2})\xi_i\overline{\xi_j}}{(\lambda_i/\lambda_j)^{1/2}-(\lambda_j/\lambda_i)^{1/2}}\\&=\sum_{i,j=1}^{n}\left(\frac{\lambda_i^{\theta}-\lambda_j^{\theta}}{\lambda_i-\lambda_j}\cdot\lambda_i^{\frac{1-\theta}{2}}\lambda_j^{\frac{1-\theta}{2}}\xi_i\overline{\xi_j}\right)\end{aligned}\tag{1}$$

for $\lambda_i>0$ and $\xi_i\in\mathbf{C}$ $(i=1,2,\dots,n)$. This computation obviously shows that $f(x)$ is positive definite if and only if the matrix

$$\left[\frac{\lambda_i^{\theta}-\lambda_j^{\theta}}{\lambda_i-\lambda_j}\cdot\lambda_i^{\frac{1-\theta}{2}}\lambda_j^{\frac{1-\theta}{2}}\right]$$

is positive (for each $n\in\mathbf{N}$ and $\lambda_1,\lambda_2,\dots,\lambda_n>0$), or equivalently so is the congruent matrix

$$\left[\frac{\lambda_i^{\theta}-\lambda_j^{\theta}}{\lambda_i-\lambda_j}\right]_{ij=1,2,\dots,n}.$$

The Fourier transform of $f(x)$ is non-negative and $f(x)$ is a typical positive definite function (by the Bochner theorem), showing the positivity of the above matrix.

The theory of operator monotone functions (see [7] for instance) neatly fits into the current picture. Let $g:[0,\infty)\to[0,\infty)$ be an operator monotone function, i.e., we have $g(A)\geqslant g(B)$

for arbitrary positive matrices A, B (of any size) with $A \geqslant B$. The operator monotonicity is known to be characterized by (one of) the following conditions:

(i) $\left[\frac{g(\lambda)-g(\lambda_j)}{\lambda_i-\lambda_j}\right]_{ij} \geqslant 0$ for each $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ (and $n \in \mathbf{N}$);
(ii) $g(x)$ extends to an analytic function on the upper half-plane $H_+ = \{z \in \mathbf{C};\ \Im z > 0\}$ satisfying $g(H_+) \subseteq H_+$.

The operator monotonicity of x^θ is well known, i.e., $\left[\frac{\lambda_i^\theta-\lambda_j^\theta}{\lambda_i-\lambda_j}\right]_{ij} \geqslant 0$ so that (1) indeed gives us an alternative proof for the positive definiteness of $\sinh(\theta x)/\sinh x$. Actually we can do much more. Namely, Horn's theorem [15] asserts that

$$\left[\frac{g(\lambda_i)-g(\lambda_j)}{\lambda_i-\lambda_j}\right]_{ij}$$

is infinitely divisible if and only if $g(z)$ is univalent (i.e., one-to-one) on H_+. This criterion guarantees the infinite divisibility of $\left[\frac{\lambda_i^\theta-\lambda_j^\theta}{\lambda_i-\lambda_j}\right]_{ij}$ (since z^θ is univalent on H_+) while almost identical computations as (1) yield

$$\sum_{i,j=1}^{n}\left(f\left(\frac{1}{2}\log\lambda_i-\frac{1}{2}\log\lambda_j\right)\right)^r\xi_i\overline{\xi_j}=\sum_{i,j=1}^{n}\left(\left(\frac{\lambda_i^\theta-\lambda_j^\theta}{\lambda_i-\lambda_j}\right)^r\lambda_i^{\frac{(1-\theta)r}{2}}\lambda_j^{\frac{(1-\theta)r}{2}}\xi_i\overline{\xi_j}\right). \tag{2}$$

Consequently, the positive definite function $f(x)=\sinh(\theta x)/\sinh x$ ($\geqslant 0$) is actually *infinitely divisible* in the sense that $f(x)^r$ is also positive definite for each $r \in (0,1)$, or equivalently, for each $r \in (0,\infty)$.

Here are some observations:

(i) If $f(x)$, $g(x)$ are positive definite, then so are the sum $f(x)+g(x)$ and the product $f(x)g(x)$.
(ii) If $f(x)$, $g(x)$ are infinitely divisible, then so is the product $f(x)g(x)$.
(iii) When a sequence $\{f_n(x)\}$ of positive definite (respectively infinitely divisible) functions is convergent, then the limit function $\lim_{n\to\infty} f_n(x)$ is also positive definite (respectively infinitely divisible).

These are obvious consequences of respective definitions, which will be freely and repeatedly used in subsequent sections.

3. A certain criterion for infinite divisibility

In this section we will present a general criterion for infinite divisibility. Our criterion is obtained by combining the key lemma below with the classical Hadamard factorization theorem (see for [6, Chapter XI, Section 3] for instance).

Lemma 1. *We assume $a > 0$ and $b \geqslant 0$. Then, the function $\frac{1+bx^2}{1+ax^2}$ is infinitely divisible if and only if $a \geqslant b$.*

Proof. The function (whose value at $x = 0$ is 1) tends to b/a as $x \to \pm\infty$. When it is infinitely divisible, it is of course positive definite, forcing $b/a \leqslant 1$. Hence, it remains to show the non-trivial converse. For this purpose we make use of the well-known integral expression

$$x^r = \frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{x}{x+\lambda} \cdot \frac{d\lambda}{\lambda^{1-r}} \quad (x \geqslant 0) \tag{3}$$

for the fractional power x^r, $0 < r < 1$, which plays an important role in the theory of operator monotone functions (see [7] for instance). Based on this formula we compute

$$\begin{aligned}
\left(\frac{1+bx^2}{1+ax^2}\right)^r &= \frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{(1+bx^2)(1+ax^2)^{-1}}{(1+bx^2)(1+ax^2)^{-1}+\lambda} \frac{d\lambda}{\lambda^{1-r}} \\
&= \frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{1+bx^2}{1+bx^2+\lambda(1+ax^2)} \frac{d\lambda}{\lambda^{1-r}} \\
&= \frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{1+bx^2}{1+\lambda+(a\lambda+b)x^2} \frac{d\lambda}{\lambda^{1-r}} \\
&= \frac{\sin(\pi r)}{\pi} \int_0^\infty \left(\frac{b}{a\lambda+b} + \frac{\lambda(a-b)}{a\lambda+b} \cdot \frac{1}{1+\lambda+(a\lambda+b)x^2}\right) \frac{d\lambda}{\lambda^{1-r}}.
\end{aligned} \tag{4}$$

Note that the integrand here is positive definite thanks to

$$\frac{1}{1+\lambda+(a\lambda+b)x^2} = \frac{1}{2\gamma(1+\lambda)} \int_{-\infty}^\infty e^{-\frac{|y|}{\gamma}} e^{ixy}\, dy \quad \text{with } \gamma = \sqrt{\frac{a\lambda+b}{1+\lambda}} \tag{5}$$

together with the positivity $\frac{b}{a\lambda+b}, \frac{\lambda(a-b)}{a\lambda+b} \geqslant 0$. Being a "superposition" of positive definite functions, $(\frac{1+bx^2}{1+ax^2})^r$ is also positive definite, i.e., $\frac{1+bx^2}{1+ax^2}$ is infinitely divisible. □

We observe

$$\frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{b}{a\lambda+b} \cdot \frac{d\lambda}{\lambda^{1-r}} = (b/a)^r$$

due to (3). After substitution of (5) into (4) we change the order of two integrals, and then we set $t = 1/\gamma$. In this way it is not so difficult to get

$$\left(\frac{1+bx^2}{1+ax^2}\right)^r = (b/a)^r + \int_{-\infty}^\infty \left(\frac{\sin(\pi r)}{\pi} \int_{1/\sqrt{a}}^{1/\sqrt{b}} e^{-t|y|} \left(\frac{1-bt^2}{at^2-1}\right)^r dt\right) e^{ixy}\, dy.$$

This expression will not be used in sequel, and details are left to the reader.

Theorem 2. *Let $f(z)$ be an entire function taking real values for the reals (the restriction to* **R** *is denoted by $f(x)$). We assume:*

(i) $f(0) > 0$ *and* $f'(0) = 0$;
(ii) *all the zeros of $f(z)$ are pure imaginary*;
(iii) *the order ρ of $f(z)$ is less than 2, i.e.,*

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} < 2 \quad \textit{with } M(r) = \max\bigl\{\bigl|f(z)\bigr|;\ |z| = r\bigr\}.$$

Under these circumstances the (real) functions $1/f(x)$ and $f(\nu x)/f(x)$ ($\nu \in [0, 1]$) are infinitely divisible.

Proof. Firstly we collect all the zeros in the upper half-plane (repeated according to multiplicity). By the assumption (ii) they are of the form $\{i\alpha_n\}_{n=1,2,\dots}$ with $0 < \alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant \cdots$. Then (thanks to the Schwarz reflection principle) all the zeros are given by $\{\pm i\alpha_n\}_{n=1,2,\dots}$. Let p be the smallest integer satisfying

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n^{p+1}} < \infty.$$

The basic property $p \leqslant \rho$ can be deduced from the Poisson–Jensen formula on distribution of zeros, and this exponent p is called the rank of $f(z)$ in [6]. The Hadamard factorization theorem enables us to factorize $f(z)$ in the following way:

$$f(z) = e^{P(z)} \prod_{n=1}^{\infty}\left(\left(1 - \frac{z}{i\alpha_n}\right)\exp\left(\frac{z}{i\alpha_n}\right)\right) \prod_{n=1}^{\infty}\left(\left(1 - \frac{z}{-i\alpha_n}\right)\exp\left(\frac{z}{-i\alpha_n}\right)\right)$$

(when $p = 1$) or

$$f(z) = e^{P(z)} \prod_{n=1}^{\infty}\left(1 - \frac{z}{i\alpha_n}\right) \prod_{n=1}^{\infty}\left(1 - \frac{z}{-i\alpha_n}\right)$$

(when $p = 0$), where the infinite product in the right-hand side is uniformly convergent on compact sets in the complex plane. Here, $P(z)$ is a polynomial of degree q and we have $\max(p, q) \leqslant \rho$. (In [6] $\max(p, q)$ is called the genus of $f(z)$.) The requirement (iii) forces $q = 0$ or $q = 1$, and we have $e^{P(z)} = f(0)e^{az}$ with some a. Observe that (when $p = 1$) the above two exponential factors cancel out, Therefore, $f(z)$ is of the form

$$f(z) = f(0)e^{az} \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{\alpha_n^2}\right).$$

Logarithmic differentiation yields

$$\frac{f'(z)}{f(z)} = a + 2z \sum_{n=0}^{\infty} \frac{\alpha_n^{-2}}{1 + z^2/\alpha_n^2},$$

and hence we must have $a = f'(0)/f(0) = 0$. This computation looks somewhat formal, but it is not. In fact, for each fixed $r > 0$ one can choose an integer n_0 large enough satisfying $|z^2/\alpha_n^2| \leqslant 1/2$ for $n \geqslant n_0$ and $|z| \leqslant r$ (due to $\alpha_n \nearrow \infty$). We thus have $|1 + z^2/\alpha_n^2| \geqslant 1/2$ and estimate

$$\sum_{n=n_0}^{\infty} \left| \frac{\alpha_n^{-2}}{1 + z^2/\alpha_n^2} \right| \leqslant 2 \sum_{n=n_0}^{\infty} \frac{1}{\alpha_n^2} < \infty.$$

From the arguments so far we have the factorization

$$\frac{f(z)}{f(0)} = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\alpha_n^2}\right),$$

showing

$$\frac{f(\nu x)}{f(x)} = \lim_{m\to\infty} \prod_{n=1}^{m} \left(\frac{1 + \nu^2 x^2/\alpha_n^2}{1 + x^2/\alpha_n^2}\right).$$

Products of infinitely divisible functions are obviously infinitely divisible. Therefore, the above finite product $\prod_{n=1}^{m}$ is infinitely divisible (for each m) thanks to Lemma 1 and so is the limit $f(\nu x)/f(x)$. $\square$

In Appendix B we will present a general result (Proposition B.1) on infinitely divisible matrices based on the Hadamard factorization, which is motivated by arguments in [1, Section 2.3]. The reasoning in the preceding proof obviously works in the following situation as well:

Corollary 3. *Let $f(z), g(z)$ be functions satisfying the conditions in Theorem 2 with the zeros $\{i\alpha_n\}_{n=1,2,\ldots}$, $\{i\beta_n\}_{n=1,2,\ldots}$, respectively, in the upper half-plane (satisfying $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots$ and $0 < \beta_1 \leqslant \beta_2 \leqslant \cdots$ with multiplicities included as before). If $\alpha_n \leqslant \beta_n$ $(n = 1, 2, \ldots)$, then the ratio $g(x)/f(x)$ is an infinitely divisible function.*

A few remarks are in order.

Remark 4.

(i) The functions

$$\cosh z, \quad \sinh z/z \quad \text{and} \quad \cosh z + s \quad \bigl(\text{with } s \in (-1, 1]\bigr)$$

are typical examples satisfying all the requirements in the theorem. Other typical examples will be also pointed out in Section 5. The well-known formulas

$$\sinh z = z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2\pi^2}\right) \quad \text{and} \quad \cosh z = \prod_{n=1}^{\infty}\left(1+\frac{z^2}{((2n-1)\pi/2)^2}\right)$$

are the Hadamard factorization for $\sinh z/z$ and $\cosh z$ (while that for the third function is to be worked out in the next section). Anyway, the theorem guarantees the infinite divisibility of the following functions:

$$\frac{\cosh(\nu x)}{\cosh x}, \quad \frac{\sinh(\nu x)}{\sinh x}, \quad \frac{x}{\sinh x}, \quad \frac{1}{\cosh x + s}, \quad \frac{\cosh(\nu x)+s}{\cosh x + s}$$

with $\nu \in [0,1]$ and $s \in (-1,1]$. In our previous studies on operator means (see [5,12–14,17] for instance) positive definiteness of relevant functions played an essential role. Factorization technique was also used in [5] to establish positive definiteness.

(ii) In [4, Theorem 3] the infinite divisibility of the function

$$\frac{x\cosh(\nu x)}{\sinh x} \quad \bigl(\nu \in [0,1/2]\bigr)$$

is proved, which can be also easily seen from Corollary 3. In fact, with $f(z) = \sinh z/z$ and $g(z) = \cosh(\nu z)$ we have

$$\alpha_n = n\pi \quad (n = 1,2,\ldots),$$
$$\beta_n = \frac{1}{\nu}\cdot\frac{(2n-1)\pi}{2} \quad (n = 1,2,\ldots),$$

and observe $\alpha_n \leqslant \beta_n$ $(n = 1,2,\ldots)$ as long as $0 < \nu \leqslant 1/2$. In the recent article [8] it is shown that the function $x\cosh(\nu x)/\sinh x$ with $\nu > 1/2$ is not positive definite.

(iii) The function $\tanh x/x$ is infinitely divisible. More generally so are the functions

$$\frac{\sinh x}{x(\cosh x + s)} \quad (-1 < s \leqslant 1).$$

Indeed, with $g(z) = \sinh z/z$ and $f(z) = \cosh z + s$ we can use Corollary 3 (see the first part of Section 4 for the zeros of the latter). A different proof for this fact is presented in the forthcoming book [2, Chapter 5].

Remark 5. A probability measure μ is said to be infinitely divisible if for each $m \in \mathbf{N}$ it can be written as the m-fold convolution product $\mu_m * \mu_m * \cdots * \mu_m$ with some probability measure μ_m. The probability distribution $\frac{a}{\pi}\cdot\frac{1}{a^2+(x-m)^2}$ (with parameters $-\infty < m < \infty$ and $a > 0$) is known as the Cauchy distribution and is a typical infinitely divisible distribution. Lemma 1 is of course closely related to this fact. An infinitely divisible probability measure plays an important role in the study of Lévy processes (see [20,21] for instance), which the author is unfortunately not so familiar with. It is known that μ is infinitely divisible if and only if the Fourier transform $\hat{\mu}$ (called a "characteristic function") admits a Lévy–Khintchine representation, i.e.,

$$\hat{\mu}(t) = \exp\left(-\frac{at^2}{2} + i\gamma t + \int_{-\infty}^{\infty} \left(e^{its} - 1 - its\, \chi_{[-1,1]}(s) \right) d\nu(s) \right)$$

with $a \geqslant 0$, $\gamma \in \mathbf{R}$ and a measure ν satisfying $\nu(\{0\}) = 0$ and $\int_{-\infty}^{\infty} \min(s^2, 1)\, d\nu(s) < \infty$ (see [21, Section 2.8] for details). However, it is practically impossible to check this criterion in our setting. Another related and useful notion is self-decomposability for probability measures (see [21, Chapter 3]): Namely, if a probability measure μ is self-decomposable (i.e., $\hat{\mu}(t)/\hat{\mu}(b^{-1}t)$ is positive definite for each $b > 1$), then it is infinitely divisible. This fact can be also used to see the infinite divisibility for some of the functions in Remark 4(i).

Remark 6. The function $\exp(-ax^2)$ (with $a > 0$) is obviously infinitely divisible, but $\exp(az^2)$ is of order 2 so that this situation is not covered in Theorem 2. Let us assume that an entire function $f(z)$ in Theorem 2 (i.e., $f(z)$ satisfies (i), (ii) and $f(\mathbf{R}) \subseteq \mathbf{R}$) is of order 2. As in the proof of Theorem 2, the zeros of $f(z)$ are of the form $\{\pm i\alpha_n\}_{n=1,2,\ldots}$ with $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots$. Let us further assume that the rank p of $f(z)$ is 0 or 1, i.e.,

$$s = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} < \infty.$$

Then, the Hadamard factorization theorem shows

$$f(z) = f(0) e^{az+bz^2} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\alpha_n^2} \right)$$

for some constants a, b (after canceling exponential factors when $p = 1$ as in the proof of Theorem 2) and with the logarithmic derivative

$$\frac{f'(z)}{f(z)} = a + 2bz + 2z \sum_{n=0}^{\infty} \frac{\alpha_n^{-2}}{1 + z^2/\alpha_n^2}.$$

Hence we get $a = 0$ again from the assumption $f'(0) = 0$. By differentiating the both sides, we observe

$$\frac{f''(z)f(z) - f'(z)^2}{f(z)^2} = 2b + 2\sum_{n=1}^{\infty} \frac{\alpha_n^{-2}}{1 + z^2/\alpha_n^2} - 4z^2 \sum_{n=1}^{\infty} \frac{\alpha_n^{-4}}{(1 + z^2/\alpha_n^2)^2},$$

showing $\frac{f''(0)}{f(0)} = 2(b + s)$ thanks to $f'(0) = 0$. Therefore, with the additional requirement

$$\frac{f''(0)}{f(0)} \geqslant 2s$$

the positivity of b is guaranteed and consequently the same conclusion as Theorem 2 is available (for $\rho = 2$ and $p \neq 2$).

4. Infinite divisibility for $(\cosh(\nu x) + s')/(\cosh x + s)$

We set

$$f_{\nu,s,s'}(x) = \frac{\cosh(\nu x) + s'}{\cosh x + s} \quad \bigl(\text{for } s, s' \in (-1, 1] \text{ and } \nu \in [0, 1]\bigr).$$

We have already known the infinite divisibility of $f_{\nu,s,s}(x)$ (see Remark 4(i)). In this section we will determine when $f_{\nu,s,s'}(x)$ is infinitely divisible. As mentioned in Section 1 this kind of information will be quite useful for investigation on generalized Heinz-type inequalities.

For $\theta \in [0, \pi)$ we set

$$g(z) = \cosh z + \cos\theta,$$

which is an entire function of order 1 due to the obvious estimate $|\cosh z| \leqslant e^{|z|}$. We will explicitly write down the Hadamard factorization for $g(z)$. (The Hadamard factorization for $\cosh z + s$ with $s \geqslant 1$ will be worked out in Appendix A, see Proposition A.1.) Let us begin with the zeros of $g(z)$. We observe

$$g(z) = 0 \quad \Longleftrightarrow \quad \cosh z = -\cos\theta = \cos(\pi - \theta)$$

so that $z = i(\pi - \theta), i(\pi + \theta)$ are zeros. They are simple zeros for $\theta \in (0, \pi)$ while $i\pi$ is a double zero for $\theta = 0$. All the zeros are obviously

$$z = i(\pi - \theta + 2n\pi), \quad i(\pi + \theta + 2n\pi) \quad (n \in \mathbf{Z}),$$

or equivalently,

$$z = \pm i(\pi - \theta + 2n\pi), \quad \pm i(\pi + \theta + 2n\pi) \quad (n = 0, 1, 2, \ldots).$$

We observe

$$\sum_{n=0}^{\infty} \frac{1}{\pi - \theta + 2n\pi} + \sum_{n=0}^{\infty} \frac{1}{\pi + \theta + 2n\pi} = \infty,$$

$$\sum_{n=0}^{\infty} \frac{1}{(\pi - \theta + 2n\pi)^{1+\varepsilon}} + \sum_{n=0}^{\infty} \frac{1}{(\pi + \theta + 2n\pi)^{1+\varepsilon}} < \infty \quad (\text{for } \varepsilon > 0),$$

showing that the exponent p (in the proof of Theorem 2) is 1. Therefore, the Hadamard factorization theorem asserts

$$g(z) = g(0)e^{az} \prod_{n=0}^{\infty} \left(\left(1 - \frac{z}{i(\pi - \theta + 2n\pi)}\right) e^{-iz/(\pi - \theta + 2n\pi)} \right)$$

$$\times \prod_{n=0}^{\infty}\left(\left(1-\frac{z}{-i(\pi-\theta+2n\pi)}\right)e^{iz/(\pi-\theta+2n\pi)}\right)$$
$$\times \prod_{n=0}^{\infty}\left(\left(1-\frac{z}{i(\pi+\theta+2n\pi)}\right)e^{-iz/(\pi+\theta+2n\pi)}\right)$$
$$\times \prod_{n=0}^{\infty}\left(\left(1-\frac{z}{-i(\pi+\theta+2n\pi)}\right)e^{iz/(\pi+\theta+2n\pi)}\right)$$

with some constant a. We can obviously rearrange involved products into the following form:

$$g(z)=g(0)e^{az}\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi-\theta+2n\pi)^2}\right)\cdot\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi+\theta+2n\pi)^2}\right).$$

We note

$$g(0)=1+\cos\theta \quad \text{and} \quad a=\frac{g'(0)}{g(0)}=0$$

(as in the proof of Theorem 2), and hence we have shown

Proposition 7. *For $\theta\in[0,\pi)$ we have the factorization*

$$\frac{\cosh z+\cos\theta}{1+\cos\theta}=\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi-\theta+2n\pi)^2}\right)\cdot\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi+\theta+2n\pi)^2}\right).$$

One should be also able to derive this factorization formula from that for $\cosh z$ (in Remark 4(i)) and the identity

$$\cosh z+\cos\theta=\cosh z+\cosh(i\theta)=2\cosh\big((z+i\theta)/2\big)\cosh\big((z-i\theta)/2\big).$$

However, the direct argument presented so far seems easier. The concrete factorization formula for $\cosh z+s$ (i.e., information on zeros) and Lemma 1 are main ingredients in the next lemma.

Lemma 8. *We assume $s,s'\in(-1,1]$ and $\nu\in[0,1]$. The function $f_{\nu,s,s'}(x)$ is infinitely divisible when the following two inequalities are satisfied*:

$$\nu\leqslant\frac{\pi-\cos^{-1}s'}{\pi-\cos^{-1}s} \quad \textit{and} \quad \nu\leqslant\frac{\pi+\cos^{-1}s'}{\pi+\cos^{-1}s}.$$

Proof. Proposition 7 shows

$$\frac{\cosh(\nu z)+s'}{1+s'}=\prod_{n=0}^{\infty}\left(1+\frac{\nu^2z^2}{(\pi-\theta'+2n\pi)^2}\right)\cdot\prod_{n=0}^{\infty}\left(1+\frac{\nu^2z^2}{(\pi+\theta'+2n\pi)^2}\right),$$
$$\frac{\cosh z+s}{1+s}=\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi-\theta+2n\pi)^2}\right)\cdot\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi+\theta+2n\pi)^2}\right)$$

with $\theta = \cos^{-1} s$ and $\theta' = \cos^{-1} s'$. Thus, thanks to Corollary 3 the ratio is infinitely divisible when

$$(\pi - \theta + 2n\pi)^{-2} \geqslant \nu^2(\pi - \theta' + 2n\pi)^{-2} \quad \text{and} \quad (\pi + \theta + 2n\pi)^{-2} \geqslant \nu^2(\pi + \theta' + 2n\pi)^{-2}$$

for each $n = 0, 1, 2, \ldots$, or equivalently,

$$\pi - \theta' + 2n\pi \geqslant \nu(\pi - \theta + 2n\pi) \quad \text{and} \quad \pi + \theta' + 2n\pi \geqslant \nu(\pi + \theta + 2n\pi)$$

for each $n = 0, 1, 2, \ldots$. We observe that as soon as these inequalities for $n = 0$ hold true then so do all the others, meaning that this condition is the same as what is stated in the lemma. □

Let us consider the following two cases:

Case $s' \geqslant s$. We have

$$\frac{\pi - \cos^{-1} s'}{\pi - \cos^{-1} s} \geqslant 1 \geqslant \nu, \tag{6}$$

i.e., the first inequality in Lemma 8 is always satisfied.

Case $s' \leqslant s$. We have

$$\frac{\pi + \cos^{-1} s'}{\pi + \cos^{-1} s} \geqslant 1 \geqslant \nu, \tag{7}$$

i.e., the second inequality in Lemma 8 is always satisfied.

The second inequality in Lemma 8 does not necessarily hold true when $s' \geqslant s$ so that we cannot use the lemma in this circumstance. Instead the following power series expansion is in rescue:

$$(1 - x)^{-r} = \sum_{n=0}^{\infty} a_n x^n \quad \text{for } r \in (0, 1) \text{ and } |x| < 1 \tag{8}$$

with the coefficients

$$a_n = \frac{r(r+1)(r+2)\cdots(r+(n-1))}{n!} \ (\geqslant 0) \quad \text{for } n = 1, 2, \ldots$$

and $a_0 = 1$.

Lemma 9. *Assume $s, s' \in (-1, 1]$. If $s' \geqslant s$, then the function $f_{\nu,s,s'}(x)$ is infinitely divisible for each $\nu \in [0, 1]$.*

Proof. We note

$$f_{\nu,s,s'}(x) = \frac{\cosh(\nu x) + s'}{\cosh x + s' - (s' - s)}$$
$$= \frac{\cosh(\nu x) + s'}{\cosh x + s'} \cdot \left(1 - \frac{s' - s}{\cosh x + s'}\right)^{-1}$$

with $0 \leqslant s' - s < \cosh x + s'$, and hence (8) yields

$$\left(f_{\nu,s,s'}(x)\right)^r = \left(\frac{\cosh(\nu x) + s'}{\cosh x + s'}\right)^r \cdot \sum_{n=0}^{\infty} \frac{(s' - s)^n a_n}{(\cosh x + s')^n}.$$

The desired infinite divisibility follows from this expression. Indeed, the first factor in the above right-hand side is positive definite thanks to Remark 4(i) while $(\cosh x + s')^{-1}$ as well as its powers are also positive definite. □

The above "power series trick" was quite useful in our previous work [4]. It will be also repeatedly used in Section 5.

Information on Fourier transforms for relevant functions is indispensable for the proof of the next lemma (Lemma 10). Here we record required formulas on Fourier transforms. They can be found in [10] (and detailed computations are presented in [18] for instance).

(i) We have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\cosh x + 1} e^{ixy}\, dx = \frac{y}{\sinh(\pi y)}.$$

Moreover, for $\nu \in [0, 1)$ we have

$$\frac{1}{2\pi} \int_{\infty}^{\infty} \frac{\cosh(\nu x)}{\cosh x + 1} e^{ixy}\, dx = \frac{y \sinh(\pi y)\cos(\pi \nu) + \nu \cosh(\pi y)\sin(\pi \nu)}{\sinh^2(\pi y) + \sin^2(\pi \nu)}.$$

(ii) For $s \in (-1, 1)$ we have

$$\frac{\sqrt{1 - s^2}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ixy}}{\cosh x + s}\, dx = \frac{\sinh(\theta y)}{\sinh(\pi y)}$$

with $\theta = \cos^{-1} s$.

(iii) For $s \in (-1, 1)$ and $\nu \in [0, 1)$ we have

$$\frac{\sqrt{1 - s^2}}{2\pi} \int_{-\infty}^{\infty} \frac{\cosh(\nu x)}{\cosh x + s} e^{ixy}\, dx$$

$$= \frac{\cos((\pi-\theta)\nu)\sinh(\pi y)\sinh(\theta y) + \sin(\pi\nu)\sin(\theta\nu)\cosh((\pi-\theta)y)}{\sinh^2(\pi y) + \sin^2(\pi\nu)}$$

with θ given in (ii). The numerator here is equal to

$$\begin{aligned}
&\big(\cos(\pi\nu)\cos(\theta\nu) + \sin(\pi\nu)\sin(\theta\nu)\big)\sinh(\pi y)\sinh(\theta y) \\
&\quad + \sin(\pi\nu)\sin(\theta\nu)\big(\cosh(\pi y)\cosh(\theta y) - \sinh(\pi y)\sinh(\theta y)\big) \\
&= \cosh(\pi y)\cosh(\theta y)\sinh(\pi y)\sinh(\theta y) + \sin(\pi\nu)\sin(\theta\nu)\cosh(\pi y)\cosh(\theta y),
\end{aligned}$$

and we easily observe that it can be also written as

$$\frac{1}{2}\big(\cos\big((\pi-\theta)\nu\big)\cosh\big((\pi+\theta)y\big) - \cos\big((\pi+\theta)\nu\big)\cosh\big((\pi-\theta)y\big)\big)$$

(by just expanding everything as above). This last expression actually appears in [10, 3.983, formula 6, p. 506].

It is possible to get the formulas in (i) from those in (iii) by letting $s \nearrow 1$, which is certainly legitimate thanks to the dominated convergence theorem.

Lemma 10. *We assume $s, s' \in (-1, 1]$ and $\nu \in [0, 1]$. If the function $f_{\nu,s,s'}(x)$ is positive definite, then we must have $\nu \leqslant \frac{\pi - \cos^{-1} s'}{\pi - \cos^{-1} s}$.*

Proof. For $\nu = 1$ we note

$$\mathcal{F} f_{1,s,s'} = \mathcal{F}\left(1 + \frac{s'-s}{\cosh x + s}\right) = \delta_0 + (s'-s)\,\mathcal{F}\left(\frac{1}{\cosh x + s}\right)$$

with the delta function δ_0 together with the Fourier transform given by either (the first part in) (i) or (ii), showing that the function is positive definite if and only if $s' \geqslant s$. Thus, in the rest we may and do assume $\nu \in [0, 1)$.

Case $s = 1$. Thanks to (i) we compute

$$\begin{aligned}
&\frac{1}{2\pi}(\mathcal{F} f_{\nu,1,s'})(y) \\
&\quad = \frac{y\sinh(\pi y)\cos(\pi\nu) + \nu\cosh(\pi y)\sin(\pi\nu)}{\sinh^2(\pi y) + \sin^2(\pi\nu)} + \frac{s' y}{\sinh(\pi y)} \\
&\quad = \frac{1}{\sinh^2(\pi y) + \sin^2(\pi\nu)} \cdot \bigg[\big(y\sinh(\pi y)\cos(\pi\nu) + \nu\cosh(\pi y)\sin(\pi\nu)\big) \\
&\qquad + \frac{s' y}{\sinh(\pi y)} \cdot \big(\sinh^2(\pi y) + \sin^2(\pi\nu)\big)\bigg] \\
&\quad = \frac{1}{\sinh^2(\pi y) + \cos^2(\pi\nu)} \cdot \bigg[\big(s' + \cos(\pi\nu)\big)y\sinh(\pi y)
\end{aligned}$$

$$+ s' \sin^2(\pi\nu) \cdot \frac{y}{\sinh(\pi y)} + \nu \sin(\pi\nu)\cosh(\pi y)\Bigg].$$

We notice that the inside of the big bracket is asymptotically equal to

$$\big(s' + \cos(\pi\nu)\big) \cdot \frac{|y|e^{\pi|y|}}{2} + \nu\sin(\pi\nu) \cdot \frac{e^{\pi|y|}}{2} \sim \big(s' + \cos(\pi\nu)\big) \cdot \frac{|y|e^{\pi|y|}}{2}$$

as $y \to \pm\infty$. Thus, if the function $f_{\nu,1,s'}(x)$ is positive definite, then the above Fourier transform is non-negative thanks to Bochner's theorem and we must have

$$s' + \cos(\pi\nu) \geqslant 0 \quad \big(\Longleftrightarrow \quad \pi\nu \leqslant \cos^{-1}(-s') = \pi - \cos^{-1} s'\big).$$

Case $s \in (-1, 1)$. Based on (ii) and (iii) (with $\theta = \cos^{-1} s$) we compute

$$\begin{aligned}
&\frac{\sqrt{1-s^2}}{2\pi}(\mathcal{F} f_{\nu,s,s'})(y) \\
&\quad = \frac{\cos((\pi-\theta)\nu)\sinh(\pi y)\sinh(\theta y) + \sin(\pi\nu)\sin(\theta\nu)\cosh((\pi-\theta)y)}{\sinh^2(\pi y) + \sin^2(\pi\nu)} \\
&\qquad + \frac{s'\sinh(\theta y)}{\sinh(\pi y)} \\
&\quad = \frac{\sinh(\theta y)}{\sinh(\pi y)(\sinh^2(\pi y) + \sin^2(\pi\nu))} \cdot \Bigg[\big(s' + \cos\big((\pi-\theta)\nu\big)\big)\sinh^2(\pi y) \\
&\qquad + s'\sin^2(\pi\nu) + \sin(\pi\nu)\sin(\theta\nu) \cdot \frac{\sinh(\pi y)\cosh((\pi-\theta)y)}{\sinh(\theta y)}\Bigg].
\end{aligned}$$

We notice that the inside of the big bracket is asymptotically equal to

$$\begin{aligned}
&\big(s' + \cos\big((\pi-\theta)\nu\big)\big) \cdot \frac{e^{2\pi|y|}}{4} + \sin(\pi\nu)\sin(\theta\nu) \cdot \frac{e^{2(\pi-\theta)|y|}}{2} \\
&\quad \sim \big(s' + \cos\big((\pi-\theta)\nu\big)\big) \cdot \frac{e^{2\pi|y|}}{4}
\end{aligned}$$

as $y \to \pm\infty$. Thus, by the same reasoning as in the previous case, the positive definiteness of $f_{\nu,s,s'}(x)$ forces

$$s' + \cos\big((\pi-\theta)\nu\big) \geqslant 0 \quad \big(\Longleftrightarrow \quad (\pi-\theta)\nu \leqslant \cos^{-1}(-s') = \pi - \cos^{-1} s'\big). \qquad \square$$

Theorem 11. *For the function*

$$\frac{\cosh(\nu x) + s'}{\cosh x + s}$$

with $s, s' \in (-1, 1]$ and $\nu \in [0, 1]$ the following three conditions are equivalent:

(i) *the function is infinitely divisible*;

(ii) *the function is positive definite*;
(iii) *the inequality*

$$\nu \leqslant \frac{\pi - \cos^{-1} s'}{\pi - \cos^{-1} s} \tag{9}$$

is satisfied.

Proof. The implication (i) ⇒ (ii) is trivial while (ii) ⇒ (iii) is exactly Lemma 10. Hence, it remains to show (iii) ⇒ (i).

Case $s' \geqslant s$. The inequality (9) is always satisfied (see (6)), and the function is indeed infinitely divisible by Lemma 9.

Case $s' \leqslant s$. The second inequality in Lemma 8 comes free (see (7)). Therefore, if (9) (i.e., the first inequality in the lemma) is satisfied, then the desired infinite divisibility is guaranteed. □

The condition (9) means that the infinite divisibility and the positive definiteness are completely governed by the location of just the "first roots" of the entire functions $\cosh(\nu z) + s'$ and $\cosh z + s$ on the imaginary axis (see the proof of Lemma 8). We note that only asymptotic behaviors of relevant Fourier transforms were needed in the preceding arguments. However, the converse of Lemma 10 can be actually proved with a little bit more effort (see [18] for details), giving rise to a direct proof for the equivalence between (ii) and (iii) in the theorem.

Remark 12. In the extreme case $\nu = 1$ the condition (9) in the theorem means $s' \geqslant s$ while this means $s' \geqslant -\sqrt{\frac{1-s}{2}}$ for $\nu = 1/2$. In fact, when $s' \in [0, 1]$, we have $\pi - \cos^{-1} s' \in [\pi/2, \pi]$ so that we always have

$$\frac{\pi - \cos^{-1} s'}{\pi - \cos^{-1} s} \geqslant \frac{\pi}{2} \cdot \frac{1}{\pi - \cos^{-1} s} \geqslant \frac{1}{2}.$$

On the other hand, when $s' \in (-1, 0]$, we have $2\cos^{-1} s' = 2\pi - \cos^{-1}(2s'^2 - 1)$ and notice

$$\begin{aligned} &\bigl(\cos^{-1}(-s) =\bigr)\, \pi - \cos^{-1} s \leqslant 2\bigl(\pi - \cos^{-1} s'\bigr) \ \bigl(= \cos^{-1}\bigl(2s'^2 - 1\bigr)\bigr) \\ \iff\quad & -s \geqslant 2s'^2 - 1 \quad\iff\quad 1 - s \geqslant 2s'^2. \end{aligned}$$

The function $\sin x/x$ is positive definite due to $2\sin x/x = \int_{-1}^{1} e^{ixy}\, dy$, and so is the square

$$\frac{\sin^2 x}{x^2} = \frac{1}{4} \int_{-\infty}^{\infty} (\chi_{[-1,1]} * \chi_{[-1,1]})(y) e^{ixy}\, dy.$$

It is known that an infinitely divisible function has no real zeros (see [20, Theorem 5.3.1, p. 108] or [21, Lemma 7.5]), and hence $\sin^2 x/x^2$ cannot be infinitely divisible. It is also known that a (non-constant) function having the Fourier transform with bounded support cannot be infinitely divisible (see [21, Corollary 24.4]). On the other hand, (non-negative) positive definite functions appearing naturally in study of operator means (see [5,12–14,17] for instance) do not have these properties. In our previous work [4] and Theorem 11 (see also Remark 4) we have actually observed that many of them are automatically infinitely divisible. It is worthwhile to investigate how general this phenomenon is.

5. Miscellaneous examples

In this section we will study other typical examples of infinitely divisible functions. We begin by recalling

$$\begin{aligned}\cosh z + \cosh(\alpha z) &= 2\cosh\big((1+\alpha)z/2\big)\cosh\big((1-\alpha)z/2\big),\\ \cosh z - \cosh(\alpha z) &= 2\sinh\big((1+\alpha)z/2\big)\sinh\big((1-\alpha)z/2\big),\\ \sinh z + \sinh(\beta z) &= 2\sinh\big((1+\beta)z/2\big)\cosh\big((1-\beta)z/2\big).\end{aligned}$$

These identities make sure that the three functions $\cosh z + \cosh(\alpha z)$, $(\cosh z - \cosh(\alpha z))/z^2$, $(\sinh z + \sinh(\beta z))/z$ (for $\alpha \in [0,1]$ and $\beta \in (-1,1]$) have zeros (only) on the imaginary axis. Therefore, by Theorem 2 (or just by combining the above three formulas and what was stated in Remark 4(i)), we conclude the following: For $\alpha \in [0,1]$ and $\beta \in (-1,1]$ the functions

$$\frac{1}{\cosh z + \cosh(\alpha z)}, \qquad \frac{z^2}{\cosh z - \cosh(\alpha z)}, \qquad \frac{z}{\sinh z + \sinh(\beta z)}$$

are infinitely divisible, and so are the functions

$$\frac{\cosh(\nu x) + \cosh(\nu\alpha x)}{\cosh x + \cosh(\alpha x)}, \qquad \frac{\cosh(\nu x) - \cosh(\nu\alpha x)}{\cosh x - \cosh(\alpha x)}, \qquad \frac{\sinh(\nu x) + \sinh(\nu\beta x)}{\sinh x + \sinh(\beta x)}$$

for each $\nu \in [0,1]$.

Here we will mainly deal with the entire function $\cosh z + s\cosh(\alpha z)$ with $\alpha \in [0,1]$ and $s \in (-1,1]$ (and related ones). To know location of zeros, we make use of the (well-known) factorization

$$\cosh(nz) = P_n(\cosh z) \quad (n = 1,2,\ldots) \tag{10}$$

with the polynomial

$$P_n(x) = 2^{n-1}\prod_{k=1}^{n}\left(x - \cos\left(\frac{(2k-1)\pi}{2n}\right)\right).$$

Indeed, let us recall

$$\begin{aligned}X^{2n}+1 &= \prod_{k=1}^{2n}\left(X - \exp\left(\frac{(2k-1)\pi i}{2n}\right)\right)\\ &= \prod_{k=1}^{n}\left(X - \exp\left(\frac{(2k-1)\pi i}{2n}\right)\right)\cdot\prod_{k=1}^{n}\left(X - \exp\left(\frac{(2(2n+1-k)-1)\pi i}{2n}\right)\right).\end{aligned}$$

Since

$$\exp\left(\frac{(2(2n+1-k)-1)\pi i}{2n}\right) = \exp\left(2\pi i - \frac{(2k-1)\pi i}{2n}\right) = \overline{\exp\left(\frac{(2k-1)\pi i}{2n}\right)},$$

the above factorization gives rise to

$$X^{2n} + 1 = \prod_{k=1}^{n}\left(X^2 - 2X\cos\left(\frac{(2k-1)\pi i}{2n}\right) + 1\right)$$

(see [10, 1.396, formula 4, p. 40] for instance). Therefore, dividing the both sides by X^n and then substituting $X = \exp z$, we conclude

$$2\cosh(nz) = \prod_{k=1}^{n}\left(2\cosh z - 2\cos\left(\frac{(2k-1)\pi}{2n}\right)\right),$$

which is exactly (10).

Lemma 13. *We assume $n, m \in \mathbf{N}$ and $n > m$. The equation $P_n(x) + sP_m(x) = 0$ has n real roots for each $s \in (-1, 1)$, and moreover all of them fall into the open interval $(-1, 1)$.*

Proof. We have to check behavior of $P_n(x)$ on the interval $[-1, 1]$. We note

$$P_n(1) = P_n(\cosh 0) = \cosh(n0) = 1,$$
$$P_n(-1) = P_n\big(\cosh(i\pi)\big) = \cosh(in\pi) = \cos(n\pi) = (-1)^n.$$

Therefore, the graph of $P_n(x)$ starts from the point $(-1, P_n(-1) = (-1)^n)$, cuts the x-axis n-times (at $\cos((2k-1)\pi/2n)$, $k = 1, 2, \ldots, n$) and ends at the point $(1, P_n(1) = 1)$. Note that local minima or maxima occur $n-1$ times somewhere in the open interval $(-1, 1)$. We claim that all of these local extrema have modulus 1. Indeed, from (10) we get

$$P_n'(\cosh x)\sinh x = n\sinh(nx).$$

Squaring the both sides, we observe

$$P_n'(\cosh x)^2\big(\cosh^2 x - 1\big) = n^2\big(\cosh^2(nx) - 1\big) = n^2\big(P_n(\cosh x)^2 - 1\big).$$

This means

$$P_n'(x)^2\big(x^2 - 1\big) = n^2\big(P_n(x)^2 - 1\big),$$

showing

$$P_n(x)' = 0 \quad\Longrightarrow\quad P_n(x) = \pm 1.$$

The discussion so far (with m instead) also shows $|sP_m(x)| \leqslant s < 1$ on $[-1, 1]$, and the assertion is now evident. $\square$

The next result is a typical example where Theorem 2 is useful even when the exact location of zeros is unknown.

Theorem 14. *For each $\alpha \in [0,1]$ and $-1 < s \leqslant 1$ the function $\cosh z + s\cosh(\alpha z)$ fulfills all the requirements in Theorem 2. In particular, all of the functions*

$$\frac{1}{\cosh x + s\cosh(\alpha x)}, \qquad \frac{\cosh(\nu x) + s\cosh(\nu\alpha x)}{\cosh x + s\cosh(\alpha x)} \quad \bigl(\nu \in [0,1]\bigr)$$

are infinitely divisible.

Proof. We may and do assume $s \in (-1,1)$ (by the discussion in the first paragraph of the section). The only thing that we have to worry about is the requirement (ii) (on the zeros) in Theorem 2. At first we assume that $\alpha = m/n$ is rational (with $m < n$). Lemma 13 says that the polynomial $P_n(x) + sP_m(x)$ (of degree n) is of the form $2^{n-1}\prod_{i=1}^{n}(x - s_i)$ with $s_i \in (-1,1)$, $i = 1,2,\ldots,n$. Therefore, we have the factorization

$$\begin{aligned}\cosh z + s\cosh(mz/n) &= P_n\bigl(\cosh(z/n)\bigr) + sP_m\bigl(\cosh(z/n)\bigr)\\ &= 2^{n-1}\prod_{i=1}^{n}\bigl(\cosh(z/n) - s_i\bigr).\end{aligned}$$

Each factor $\cosh(z/n) - s_i$ admitting only pure imaginary zeros, so does the product $\cosh z + s\cosh(mz/n)$.

For a general $\alpha \in [0,1]$ one chooses a sequence $\{r_i\}_{i=1,2,\ldots}$ of rationals tending to α. Then, the sequence $\{\cosh z + s\cosh(r_i z)\}_{i=1,2,\ldots}$ of entire functions converges uniformly to $\cosh z + s\cosh(\alpha z)$ on each compact set so that the limit function $\cosh z + s\cosh(\alpha z)$ admits only pure imaginary zeros thanks to the first half of the proof and Hurwitz's theorem (see [6, p. 152] for instance). □

The author is unable to determine what happens for $s > 1$. On the other hand, in [3, Theorem 1.2] it was shown that the matrices

$$\left[\frac{1}{\lambda_i^3 + \lambda_j^3 + s(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)}\right]_{i,j=1,2,\ldots,n} \qquad (\text{for each } s > -1)$$

are always positive for each $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ (and for each $n \in \mathbf{N}$), i.e., the function

$$\frac{1}{\cosh t + s\cosh(t/3)}$$

is positive definite for each $s > -1$ (by computations analogous to (1)). This positive definiteness remains valid for $\cosh(t/2)$ instead of $\cosh(t/3)$ [18].

Formulas for the Fourier transforms of functions of the form $(\cosh x + s)^{-1}$ were needed in Section 4 (especially for the proof of Lemma 10). One standard way to derive them is the so-called method of residues. Note that this method cannot be used for evaluating the integral

$$\int_{-\infty}^{\infty} \frac{e^{ixy}\,dx}{\cosh x + s\cosh(\alpha x)}$$

(for general α). Indeed, one cannot write down zeros for $\cosh z + s\cosh(\alpha z)$ explicitly so that residues of the integrand (as a complex function) are not computable. Therefore, it seems impossible (at least to the author) to confirm even positive definiteness of the functions in question via Fourier transform approach. Anyway, results so far can be combined with the power series expansion trick (used in the proof of Lemma 9) to show the next result.

Proposition 15. *We assume* $\alpha, \nu \in [0,1]$ *and* $s, s' \in (-1, 1]$.

(i) *The function* $\frac{\cosh(\nu x)}{\cosh x + s\cosh(\alpha x)}$ *is infinitely divisible as long as* $\nu \leqslant \frac{1+\alpha}{2}$.
(ii) *The function* $\frac{\cosh(\nu x) + s'\cosh(\nu\alpha x)}{\cosh x + s\cosh(\alpha x)}$ *is infinitely divisible as long as* $s \leqslant s'$.

Proof. To see (i) for $s = 1$, we note

$$\frac{\cosh(\nu x)}{\cosh x + \cosh(\alpha x)} = \frac{\cosh(\nu x)}{2\cosh((1+\alpha)x/2)\cosh((1-\alpha)x/2)},$$

which is infinitely divisible due to the assumption $\nu \leqslant (1+\alpha)/2$ (see Remark 4(i)). We then assume $s \in (-1, 1)$ and compute

$$\begin{aligned}\frac{\cosh(\nu x)}{\cosh x + s\cosh(\alpha x)} &= \frac{\cosh(\nu x)}{\cosh x + \cosh(\alpha x) - (1-s)\cosh(\alpha x)} \\ &= \frac{\cosh(\nu x)}{\cosh x + \cosh(\alpha x)} \cdot \left(1 - \frac{(1-s)\cosh(\alpha x)}{\cosh x + \cosh(\alpha x)}\right)^{-1}\end{aligned}$$

with $0 \leqslant (1-s)\cosh(\alpha x) < \cosh x + \cosh(\alpha x)$. Thus, by making use of the power series expansion (8) we get

$$\left(\frac{\cosh(\nu x)}{\cosh x + s\cosh(\alpha x)}\right)^r = \left(\frac{\cosh(\nu x)}{\cosh x + \cosh(\alpha x)}\right)^r \cdot \sum_{n=0}^{\infty} \frac{a_n (1-s)^n \cosh^n(\alpha x)}{(\cosh x + \cosh(\alpha x))^n}$$

for each $r \in (0,1)$. The first rth power in the right-hand side is positive definite by the first part of the proof while the obvious fact $\alpha \leqslant (1+\alpha)/2$ guarantees the positive definiteness (indeed the infinite divisibility) of each $\cosh^n(\alpha x)/(\cosh x + \cosh(\alpha x))^n$, showing (i).

To prove (ii), we need to observe

$$\frac{\cosh(\nu x) + s'\cosh(\nu\alpha x)}{\cosh x + s\cosh(\alpha x)}$$

$$= \frac{\cosh(\nu x) + s' \cosh(\nu\alpha x)}{\cosh x + s' \cosh(\alpha x)} \cdot \left(1 - \frac{(s' - s)\cosh(\alpha x)}{\cosh x + s' \cosh(\alpha x)}\right)^{-1}.$$

Then, the result follows from Theorem 14 and (i) together with the usual trick based on (8) repeatedly used so far. □

The condition $\nu \leqslant \frac{1+\alpha}{2}$ in (i) does not involve the parameter s, and is not an optimal one. For instance, for $s \in (-1, 0]$ the function in question is infinitely divisible always (i.e., for each $\nu \in [0, 1]$) thanks to (ii). However, some kind of restriction is unavoidable. Indeed, Theorem 11 says that for $\alpha = 0$ and $s = 1$ it is infinitely divisible if and only if $\nu \leqslant 1/2$.

Proposition 16. *We assume* $\alpha \in [0, 1]$. *The functions*

$$\frac{x}{\sinh x + s \sinh(\alpha x)}, \qquad \frac{\sinh(\nu x)}{\sinh x + s \sinh(\alpha x)} \quad \big(0 < \nu \leqslant (1+\alpha)/2\big)$$

are infinitely divisible.

Proof. We note

$$\frac{\sinh(\nu x)}{\sinh x + \sinh(\alpha x)} = \frac{\sinh(\nu x)}{2\sinh((1+\alpha)x/2)\cosh((1-\alpha)x/2)},$$

$$\frac{\sinh(\nu x)}{\sinh x + s\sinh(\alpha x)} = \frac{\sinh(\nu x)}{\sinh x + \sinh(\alpha x) - (1-s)\sinh(\alpha x)}$$

$$= \frac{\sinh(\nu x)}{\sinh x + \sinh(\alpha x)} \cdot \left(1 - \frac{(1-s)\sinh(\alpha x)}{\sinh x + \sinh(\alpha x)}\right)^{-1}.$$

These together with arguments as in the proof of Proposition 15(i) yield the infinite divisibility of the second function. The infinite divisibility of the first can be obtained by similar arguments or from the obvious fact

$$\frac{x}{\sinh x + \sinh(\alpha x)} = \lim_{\nu \searrow 0}\left(\frac{1}{\nu} \cdot \frac{\sinh(\nu x)}{\sinh x + \sinh(\alpha x)}\right). \qquad \square$$

Appendix A

We will explicitly write down the Hadamard factorization for the function $\cosh z + s$ with $s \geqslant 1$. After some trials one is convinced that the following parameterization is more convenient:

$$f(z) = \cosh(\pi z) + \cosh(\pi a) \quad (\text{with } a \in \mathbf{R}).$$

It is elementary to see that the zeros of $f(z)$ are

$$z = \pm a + i + 2ni, \quad n \in \mathbf{Z},$$

or equivalently,

$$\begin{cases} a + (2n+1)\,i \\ a - (2n+1)\,i \end{cases} \quad (\text{for } n = 0, 1, \ldots) \quad \text{and} \quad \begin{cases} -a + (2n+1)\,i \\ -a - (2n+1)\,i \end{cases} \quad (\text{for } n = 0, 1, \ldots).$$

They are all simple zeros (unless $a = 0$), and the rank of $f(z)$ is obviously 1. We note

$$\frac{1}{a \pm (2n+1)\,i} = \frac{a \mp (2n+1)\,i}{a^2 + (2n+1)^2} \quad \text{and} \quad \frac{1}{-a \pm (2n+1)\,i} = \frac{-a \mp (2n+1)\,i}{a^2 + (2n+1)^2}.$$

Based on these facts we easily compute

$$\begin{aligned}
&\left(1 - \frac{z}{a + (2n+1)i}\right) e^{z/(a+(2n+1)i)} \cdot \left(1 - \frac{z}{a - (2n+1)i}\right) e^{z/(a-(2n+1)i)} \\
&\quad = \left(1 + \frac{z^2 - 2az}{a^2 + (2n+1)^2}\right) e^{2az/(a^2+(2n+1)^2)}, \\
&\left(1 - \frac{z}{-a + (2n+1)i}\right) e^{z/(-a+(2n+1)i)} \cdot \left(1 - \frac{z}{-a - (2n+1)i}\right) e^{z/(-a-(2n+1)i)} \\
&\quad = \left(1 + \frac{z^2 + 2az}{a^2 + (2n+1)^2}\right) e^{-2az/(a^2+(2n+1)^2)}.
\end{aligned}$$

Thus, the Hadamard factorization is

$$\begin{aligned}
f(z) = f(0)e^{bz} &\prod_{n=0}^{\infty} \left(\left(1 + \frac{z^2 - 2az}{a^2 + (2n+1)^2}\right) e^{2az/(a^2+(2n+1)^2)} \right) \\
&\times \prod_{n=0}^{\infty} \left(\left(1 + \frac{z^2 + 2az}{a^2 + (2n+1)^2}\right) e^{-2az/(a^2+(2n+1)^2)} \right)
\end{aligned}$$

with some b. But, we get $b = 0$ again due to $f'(0) = 0$. Therefore, by canceling two exponential factors, we have shown the following factorization:

Proposition A.1. *For a real we have*

$$\begin{aligned}
&\frac{\cosh(\pi z) + \cosh(\pi a)}{1 + \cosh(\pi a)} = \prod_{n=0}^{\infty} \left(1 + \frac{z^2 + 2az}{a^2 + (2n+1)^2}\right) \cdot \prod_{n=0}^{\infty} \left(1 + \frac{z^2 - 2az}{a^2 + (2n+1)^2}\right) \\
&\quad \left(= \prod_{n=0}^{\infty} \left(1 + \frac{z^4 + 2(2n+1+a)(2n+1-a)z^2}{(a^2 + (2n+1)^2)^2}\right) \right).
\end{aligned}$$

The appearance of $\pm 2az$ makes it impossible to use Lemma 1 (unless $a = 0$). In fact, the function in question is not positive definite (as was pointed out in [5]).

Appendix B

The Pascal matrix

$$\left[\frac{(i+j)!}{i!\,j!} \right]_{i,j=0,1,\ldots,n-1}$$

and its generalizations $P = [p_{ij}]_{i,j=1,2,\ldots,n}$ with entries

$$p_{ij} = \frac{\Gamma(\lambda_i + \lambda_j + 1)}{\Gamma(\lambda_i + 1)\Gamma(\lambda_j + 1)} \quad \text{(with arbitrary positive reals } \lambda_1, \lambda_2, \ldots, \lambda_n\text{)}$$

were studied in [1], and based on the classical Gauss formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)\cdots(z+n)} \quad (z \neq 0, -1, -2, \ldots)$$

the infinite divisibility of P was proved, i.e., $P^{\circ r} = [p_{ij}^r] \geqslant 0$ for each $r \in (0, 1)$ (see Section 2.1). The reasoning in [1] can be formulated in the following general form, and $1/\Gamma(z+1)$ is indeed a typical example satisfying all the requirements below:

Proposition B.1. *Let $f(z)$ be an entire function taking real values for the reals. We assume* (i) $f(0) > 0$, (ii) *all the zeros of $f(z)$ are real and negative, and* (iii) $\rho < 2$. *In this case, we have $f(x) > 0$ for $x \geqslant 0$, and for each $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ the matrix*

$$\left[\frac{f(\lambda_i)f(\lambda_j)}{f(\lambda_i + \lambda_j)}\right]_{i,j=1,2,\ldots,n}$$

is infinitely divisible.

Proof. Let $\{-\alpha_n\}_{n=1,2,\ldots}$ be the zeros of $f(z)$ with $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots$ (repeated according to multiplicity). The rank p of $f(z)$ is 0 or 1. For instance, when $p = 1$, the Hadamard factorization is

$$f(z) = f(0)e^{az} \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{\alpha_n}\right)e^{-z/\alpha_n}\right)$$

(with $a = f'(0)/f(0)$), showing $f(x) > 0$ for $x \geqslant 0$.

Since the maps $z \to e^{az}$ and $z \to e^{-z/\alpha_n}$ are multiplicative (and infinite divisibility is preserved by taking products), it suffices to show the infinite divisibility of the matrix P with entries

$$p_{ij} = \frac{(1 + \lambda_i/\alpha_n)(1 + \lambda_j/\alpha_n)}{1 + (\lambda_i + \lambda_j)/\alpha_n}.$$

Note that P is congruent to the matrix with entries

$$\frac{1}{1 + (\lambda_i + \lambda_j)/\alpha_n} = \left(\frac{\alpha_n}{(\lambda_i + \alpha_n/2) + (\lambda_j + \alpha_n/2)}\right)$$

so that the desired result follows from the infinite divisibility of the Cauchy matrix $[(\lambda_i + \lambda_j)^{-1}]$ (see Section 1). $\square$

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